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ABSTRACT

A discussion is presented concerning variables associated with children's representation of certain addition and subtraction problems. Problem components are classified as physical, pictorial, sensual, verbal-spoken, verbal-written, symbolic, or internal. Problems are discussed as abstract (symbolic) or nonsymbolic. Three problem components are identified and discussed: (1) measurable entities, (2) contextual relationships or actions on entities, and (3) characteristics of the missing number. The second component is further broken down into joining, separating, equalizing, part-part-whole, and comparison. Problem structure is also discussed. (MK)

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Theoretical Paper No. 74

Conceptual Paper No. 4

YOUNG CHILDREN'S REPRESENTATION OF ADDITION AND SUBTRACTION PROBLEMS

by

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Report from the Project on
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ABSTRACT

This paper presents a discussion of some variables associated with children's representation of certain addition and subtraction problems. Representation is defined as the change of modality of a problem component. The possible modalities that can be used in problem representation are physical, pictorial, sensual, verbal-spoken, verbal-written, symbolic, and internal.

Three components of addition or subtraction problems are identified. These are the two known numbers or measurable sets given, the relationship or action connecting the three numbers or sets of the problem, and the question or context that identifies the third missing number. These components are descriptive of the two major classes of addition and subtraction problems, namely symbolic and non-symbolic.

Non-symbolic problems can be further described along three major dimensions. These are whether or not there is a set-inclusion relationship among the three measurable entities of the problem, whether there is an active or static relationship, or whether the action or relationship is ordered in the sense of being incremental or decremental. A number of examples of problems fitting within the various dimensions are presented.

Finally, the so-called structure of problems is discussed, both from a mathematical and psychological point of view. It is argued that the psychological structure imposed by a child is a function of several variables, including problem type, number size, semantic considerations, amount and type of formal instruction received, and stage of cognitive development.

Introduction

One of the major aims of mathematical instruction is to enable the student to acquire knowledge and skills necessary for solving problems of many types. These skills include simple computing, constructing a proof in geometry, solving a system of equations, evaluating line integrals, and so forth. Since 1977 the Mathematics Work Group of the Wisconsin Research and Development Center for Individualized Schooling has been engaged in a program of research investigating how young school-age children acquire the skills needed to solve certain kinds of addition and subtraction problems. It is the purpose of this paper to set forth some ideas and definitions related to the skill of representation.

Representation is frequently described as the translation or modeling of a problem by an appropriate mathematical symbolism. This might mean constructing a chart or graph, writing a differential equation, or using some formula. For this paper, a more general definition of representation will be advanced. Although the definition may well apply to a larger class of problems, the reader is reminded that the problems being discussed here are restricted to those involving addition and subtraction. It is also necessary to recognize that simply talking about a "problem" is probably not specific enough.

Addition and subtraction problems have clearly identifiable components--the known numerical quantities, the relationship and/or

operation connecting those quantities, and the context of the unknown quantity and how it is related to the other two quantities. It makes more sense to deal with the components of a problem when defining representation. The components of a problem can be present within a number of different modalities or modes, either when the problem is initially presented or during its varying stages of being solved. The different modalities include physical, pictorial, sensual, verbal-spoken, verbal-written, symbolic, or internal--or some combination of these.

The skill or process of representation is defined as the change from one modality to another. Several examples may be useful in clarifying the definition.

Example 1. While using a set of physical objects, a child counts the objects, one-by-one, saying, "one, two three, ...". Here, the modality is being changed from physical to verbal-spoken.

Example 2. As part of a problem presentation, a picture of six blackbirds sitting on a tree limb is shown. At the same time, the verbal-written statement, "Two birds fly away," is made. The child represents the action of flying away by using his or her pencil to cross out two of the pictured birds. The representation involves a change from the verbal-written modality to the pictorial modality.

Example 3. A part of an addition problem is presented as four taps on a drum. The child listens to the drum beats and then writes the symbol "4." The representation involves a change from the sensual (in this case, aural) modality to the symbolic modality.

It should not be assumed that representation is the only skill that children use in a problem-solving process. Another important skill is transformation. When children remain within a given modality

and change the form of the problem (or one of its components), they are using a transformation skill.

> Example 4. A child has been presented with a problem in the verbal-spoken mode. Suppose it has an additive structure in the sense that a set of 5 things is to be joined to a set of 9 things. The child begins by representing the 5 things with 5 counting chips and the 9 things with 9 counting chips, setting the 9 chips out separately from the 5 chips. The child then moves 1 chip away from the pile of 5, adjacent to the pile of 9 chips. The child has remained within the physical modality, but has changed the form from 9 and 5 (or 9 + 5, if you like) to 10 and 4 (or 10 + 4).

As children attempt to solve a problem they are likely to use both representation and transformation skills. In fact, we assume that the problem-solving process involves some finite sequence of representations and/or transformations that begins with the presentation of the original problem and ends with the final solution, which may or may not be correct. The number of different possible sequences for a particular problem is also probably finite, althought quite large. Identification of some of these solution sequences and the reasons why an individual uses one sequence as opposed to another is one of the aims of the research program of the Mathematics Work Group. Transformations will not be discussed further in this paper. For a more complete characterization, see Romberg, Pimm and Small (Note 1).

Since this paper is concerned with representation of addition and subtraction problems, it is necessary to briefly consider the general structure of those problems. A suitable mathematical model for addition/subtraction problems is the sentence $a + b = c$ or

$a - b = c$, where two of the numbers represented by a, b, and c are known and the third is missing or unknown. Three identifiable components of the problem can be considered. First, the two entities associated with the known numbers may be the numbers themselves as in a purely symbolic situation, or two sets, objects, events, distances, or whatever for which the two numbers are the measure. Second, there is the relationship or action that connects the entities or numbers, known and unknown. And finally, there is the question within the problem that seeks the identification of the third, unknown number. It is hypothesized that the interpretation of the second and third components significantly influences the particular sequence of representations and/or transformations used by the problem solver. On the other hand, the first and second components are amenable to representation by the problem solver. These components will be discussed in detail in a later section of this paper. But first we turn to a discussion of the various types of problem modalities.

Problem Component Modalities

Recall that a representation is defined as a change from one modality to another. A modality or mode is simply an embodiment of a problem component. Although it is convenient to list and characterize each type of modality separately, it is recognized that in actual problem-solving situations these various embodiments may not exist independently. This is particularly true during the initial problem

presentation to the student where a language or verbal factor is almost always present.

It is important to remember that the initial modality occurs during the problem presentation, which is generally external to the child. However, it is not safe to assume that just one act of representation will take place. A child may change modalities of a problem component several times before the problem-solving process is ended.

As the various modalities are discussed, examples will be presented. In certain instances, an example of the modality by itself will not be as meaningful as when the example is given within the context of a representation taking place. Hopefully, it will be clear which modality is being discussed.

Physical

The physical mode involves the use of concrete, manipulatable objects.

> Example 5. Consider the following problem, initially presented to a child in the verbal-spoken mode. "Sue has 11 candies. She gave 4 candies to Ralph. How many candies does Sue have left?" The child uses plastic cubes to stand for the candies, setting out 11 cubes upon which some further physical action may or may not take place. The quantitative component is now embodied by the physical objects.

In certain instances, physical objects can be part of the initial problem stimulus. For example, in studies done by Steffe (1970) and Le Blanc (1971), more realistic objects such as toy cars or plastic

dogs pasted to cardboard were used as part of a problem situation. See also Carpenter, Hiebert, and Moser (Note 2).

Included in this category is the use of fingers. Perhaps the most common use of fingers occurs when children let their fingers take the place of a sequence of spoken counting words, in which case each finger in turn is associated with a specific spoken numeral.

Another aspect of the physical modality that deserves particular mention is physical action performed on some object or set of objects. In this case, the physical act is the representation of some described action given in a verbal modality. For instance, in example 5 in which 4 candies are given away from a set of 11 original candies, the described action of giving away may be represented by the physical removal of 4 plastic cubes from the set of 11 cubes that had been put out.

Pictorial

As its name implies, the pictorial modality of a problem component involves the use of pictures or tally marks. A common example of this modality is the printed materials used during instruction in early grades. Pictures of sets of birds, coins, children, etc. are often used as part of an initial problem presentation. Other pictorial conventions include marks or arrows which suggest some action such as movement of part of a pictured set away from the rest of the set.

A child's use of the pictorial mode may come as a response to a problem posed in some other mode. Crude pictures are constructed

which may or may not resemble the entities described in the original problem. Other manifestations of this modality are lines or marks that children use to represent actions or relationships contained within the problem situation. For example, if a giving-away or going-away situation is involved, the child may cross out some pictured objects. Another device is lines drawn between members of sets of objects to indicate a one-to-one pairing or matching between those members.

Another type of pictorial modality that can be used is the graph. This is somewhat more abstract than simple pictures because squares on a bar graph or points on a line graph are used to stand for single entities or sets of entities.

Sensual

This category includes distinct signals that impinge upon human senses other then recognizable language. Some examples are aural signals such as claps of the hands, or musical sounds from drum beats or notes on a piano, visual stimuli such as turning lights on and off, and pressure signals such as touching a child's shoulder a number of times. Though not as commonly used as other modalities, it is possible to imagine a teacher asking a class to tell how many more drum beats were sounded than tinklings of a triangle.

Within this modality, an operation or action of putting together a succession of sensual signals may be indicated by nothing more than a prolonged pause occurring within the sequence of the signal transmissions.

Verbal-spoken

This commonly used modality is recognized by its use of standard language forms that are produced by human voice or electronic reproductions and interpreted by the human ear. An obvious example occurs when a teacher reads a problem to a child or a class of children, or when a child reads aloud all or part of a problem. Another important example occurs when a child recites one or more number words that constitute a counting sequence.

Verbal-written

Here the standard language forms occur in print. Perhaps the overwhelming majority of instances in which this modality appears is the initial problem presentation. In other words, printed problems or printed instructions that accompany pictorial or symbolic modalities are given to the child on workbook or textbook pages or on duplicated sheets that a teacher prepares. It is relatively rare when a child initiates this modality.

Symbolic

As indicated by its name, the symbolic mode of representation involves the use of numerical, operational (+ or -), and relational (=, ≠, >, <) symbols. When the symbolic mode occurs as the initial problem presentation, the symbols are juxtaposed into what is known as a mathematical sentence. The generic form can be horizontal, such as $a + b = c$ or $a - b = c$, or vertical, such as $\frac{a}{+b}{c}$ or $\frac{a}{-b}{c}$, in which case the horiztonal vinculum takes the place of the "=" symbol in the horizontal form.

When the symbolic mode occurs as the result of the child initiating the representation, the form need not be a mathematical sentence. Often, a child may simply write a single numeral.

If a written numeral appears within the context of an otherwise verbal-written modality, we will agree that it is part of that verbal modality. In other words, the symbolic modality should be essentially purely symbolic, apart from some printed instructions such as "Solve."

Internal

This type of modality is the least amenable to careful and precise definition. It might best be characterized as some sort of mental imagery as used by the problem solver. Although not directly observable, this writer and his colleagues have, on certain occasions while interviewing young children, heard a response to the effect, "I can see those (entities) in my head and I grouped them so that there were x (a number) here and the rest were over there." Often the internal visualization concerns numbers. Whether the image is a symbolic numeral or a definite collection of the appropriate size is uncertain. Yet, we have had children report these mental activities.

It is not the intent to include within this category the vast assortment of mental and intellectual processes particular to the human mind. The many human mental processes such as information processing, storage and retrieval, decision making, hypothesis generation or testing, speaking, reading, seeing, hearing, and reasoning are not considered to be problem components.

Characterization of Addition and Subtraction Problems

Within this second major section of the paper, we turn to addition and subtraction problems presented to young children. These problems can be grouped into two large categories. One category is the purely mathematical presentation of symbolic sentences in either horizontal or vertical form. The other is the set of "real" problems; the quantities in question are measures of the entities described in the problem. Included in this second category are the so-called "story problems" presented in textbooks. It can be argued that the latter category relates more to a conceptual understanding of instructional objectives, while the former is more closely connected to the skill of instruction. Such a dichotomy is perhaps too simplistic a notion since both types of problems and their related solution processes are important to the growth and development of mathematically literate persons. It is not an easy matter to distinguish between skills and understandings, nor is it easy to decide where one begins and the other leaves off.

Three components of a problem have been identified. First, there are the two known numbers or the entities associated with, or representable by, these two numbers. The entities may be sets, objects, events, distances, durations, movements, or any other measurable thing. Second, there is an operation, relationship, or action that connects together the three numbers or entities in some way. Third, there is the question or situation within the problem that seeks the identification of the unknown number or the measure of the third entity.

Abstract or Symbolic Problems

The abstract presentation of horizontally written addition and subtraction problems is often called the solution of open sentences. The typical open sentence usually has only one of the three numbers that is unknown. That is symbolized by the "box" as in the open sentence $14 - 9 = \square$. It is also possible to represent the missing number by a letter, in which case the open sentence looks more like an equation from an algebra course.

The generic forms of a simple addition or subtraction sentence are $a + b = c$ and $a - b = c$. Taking note of the fact that the equality relation has a symmetric property, it is possible to have a horizontal sentence that the form of $c = a \pm b$. This latter is an example of a right-hand sentence, while the more common forms are called left-hand sentences. Table 1 summarizes the different forms that open sentences may take.

For open sentences, the first problem component is easily recognized by the two given numbers. The second component is the operational symbol present in the given sentences. As Table 1 suggests, it is either the + (addition) symbol or the - (subtraction) symbol. It is important to note that the term "addition sentence" refers to the given symbol and not to the operation by which one may actually solve the sentence.

The third problem component for the abstract, symbolic presentation is identified by the position of the "box" standing for the missing number. When it is in the "c" position, and the sentence takes the form

Table 1

Forms of Open Addition and Subtraction Sentences

Left-hand	Right-hand
Addition	
1. $a + b = \square$	1.′ $\square = a + b$
2. $a + \square = c$	2.′ $c = a + \square$
3. $\square + b = c$	3.′ $c = \square + b$
Subtraction	
4. $a - b = \square$	4.′ $\square = a - b$
5. $a - \square = c$	5.′ $c = a - \square$
6. $\square - b = c$	6.′ $c = \square - b$

$a \pm b = \square$, the open sentence is said to be in canonical form. If the missing number is in the "a" or "b" position, the open sentence is then in noncanonical form.

As Weaver (1971) points out, sentences 1, 1', 6, and 6' (Table 1) are such that substitution of any whole number for the variables a, b, and c will produce an open sentence that has a solution within the set of whole numbers. However, for the remaining eight sentence types, certain choices of whole numbers will produce an open sentence that has no solution in the set of whole numbers. For example, a specific instance of this second type is $8 + \square = 3$. Therefore, there are 20 distinguishable problem types of open sentences, 12 that have a solution in the set of whole numbers and 8 that do not.

The presentation of a totally symbolic addition/subtraction problem is a frequent occurrence in primary mathematics instructional programs. The open sentences are often accompanied by a short set of directions. In general, the instructions may be "Solve" or "Solve the open sentence," particularly when the sentence is horizontal. For the canonical addition sentence, the directions are "Add," "Find the sum," or (less frequently) "Find the total." For the canonical subtraction sentence, the directions are usually "Subtract" or "Find the difference." It should also be noted that noncanonical vertical presentations of addition and subtraction are relatively rare in most instructional programs.

Nonsymbolic Problems

A suitable mathematical representation for the simple addition or subtraction problem is the sentence $a + b = c$ or $a - b = c$ in which one of the numbers a, b, or c is unknown and the other two are known. This mathematical model is presented here in order to facilitate explanation. There is no assumption that children will write such sentences as they attempt to solve the given problem.

The three components of an addition or subtraction problem have been identified. For the symbolic problems, these three components are easily identifiable. However, for the nonsymbolic, they are less readily identified. For that reason, they will be discussed in turn in the sections that follow.

First problem component: Measurable entities. A major distinction may be drawn between problems involving quantities that are discrete entities and problems involving measures of continuous quantity, like weight, length, volume, or age. In the first case, it is possible to represent the elements in the given problem set (apples, children, or whatever) by counting out the appropriate number of elements to make an equivalent set. There is a one-to-one correspondence between the problem set and the constructed set so that one can actually think of each element in the constructed set as representing an element in the problem set.

For continuous measures, however, any set that might be constructed would represent the quantity in a very different sense. There

would not be a one-to-one correspondence, because there are no identifiable elements in this type of problem set. Thus, for continuous quantity problems, the constructed set would represent the number assigned to the attribute in the problem, but would not represent the attribute directly. Needless to say, continuous quantities present potentially more complex problem situations. It is also possible that continuous measures that are not directly observable such as age or weight are more complex than measures such as length or area that tend to be more easily discerned on a visual basis. For simplicity, the examples that follow rely on problems involving discrete quantities. Parallel problems could certainly be constructed involving continuous quantities.

If the initial problem presentation is in the physical mode, it is assumed that the child would have to initiate the measuring process by assigning a number to the entities described in that problem. If the particular measurement skill required is inaccessible, then the solution process could not be carried to its end. If the verbal modes were used, then it is likely that the actual measurements of the entities would be given as part of the problem presentation. It is not known whether differentiation of attribute type--discrete, observable continuous, or nonobservable continuous--would initiate different sequences of problem-solving behaviors in terms of the representation used. Further empirical investigation is needed to determine possible answers to such a question.

Second problem component: Contextual relationships or actions on entities. This second component is of major significance. Together with the first component, it may be represented by a variety of modalities. But what is perhaps even more important, it provides, along with the third component, the structure of the problem itself. As indicated in the introduction to this paper, the interpretation of this problem structure by the child is probably the single most positive determinant of the sequence of representations and transformations that will be used to solve the problem.

The characterization of this second component -- the relationship or action on the measurable entities -- has three dimensions. The first is the presence or absence of a set inclusion relationship. Next, it is possible to distinguish between a clearly implied action involving the entities as opposed to a static relationship. Finally, an order relationship may or may not be present. We shall amplify on these dimensions in the succeeding paragraphs. In the following discussion, A, B, and C will represent the sets or entities to which the numbers a, b, and c can be assigned. Further, a set or entity A' will be related to A because the measure of A' = a; this will also be true for any set or entity B' or C'.

One may consider the set inclusion or set-subset relationship between the entities. One relationship that occurs within addition/subtraction situations is when two of the entities are disjoint, but are subsets of the third entity. Thinking of the mathematical models $a + b = c$ and $a - b = c$, these set-subset relationships can be

represented by $A \cup B = C$ and $A = B \cup C$. Another expression for the latter is the set theoretic difference $A / B = C$ where $A \supseteq B$. A second, identifiable relationship that occurs within addition/subtraction situations is characterized by the fact that one of the entities may be a subset of, or included in, a second entity. But, in this case, the third entity is not included in the second or the first. In set notation, this relationship would be $A \subseteq C$, but $B \not\subseteq C$ or $B \not\subseteq A$. Similar relationships could be expressed by suitable interchanges of the symbols A, B, and C.

The second dimension is identified with some potential action or movement involving the entities. Consider an explicit statement of action or movement which causes a change in size or position of the entities. Steffe (1970) and Le Blanc (1971) talk of such situations as having a described transformation; Greeno (Note 3) describes this as the Cause/Change schema; Nesher and Katriel (Note 4) call this the Dynamic description. On the other hand, there can be a static situation involving the entities.

The third dimension involving the contextual relationship among the measurable problem entities is best described as an order relationship. In the static relationship among the entities, there is the notion that one entity is larger or smaller than another. Where action is implied in the problem, that action results in something being made larger (increased) or being made smaller (decreased).

Thus, each of these three dimensions can be dichotomized. It is helpful to visualize this overall characterization of problem types by means of a three-dimensional solid that represents a two-by two-by-two matrix. Such a solid is depicted in Figure 1.

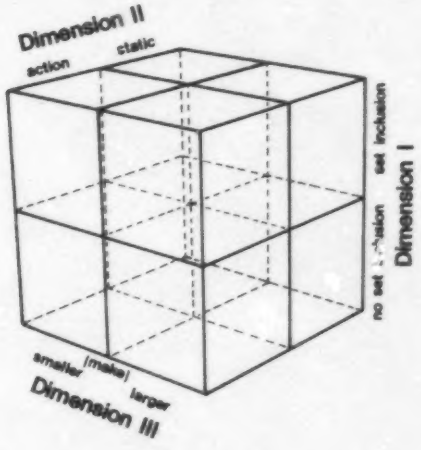

Figure 1. Schematic diagram of characteristics of addition/subtraction non-symbolic problems.

We now turn to each of the cells in the three-dimensional matrix. Most can be exemplified by problem types that are generally well-known in the literature. Although the third basic problem component -- the nature or position of the missing third number -- has not been discussed, we will provide examples in which the missing number appears in various positions. To help clarify this and to relate it to the previous consideration of symbolic problems, the open sentence representation of the sample problems will be provided following each example.

Joining

This problem situation arises frequently in early mathematics instruction because it is generally easy to understand and it tends to be a familiar situation for young children. Joining is the process of actively putting together a set or entity B with a set or entity A to form a new single set or entity C. A is made larger or increased by B so that the union has measure c. Both A and B are subsets of C. It is often the case that once the union or join of A and B is carried out to form C, original sets or entities A and B are no longer distinguishable from each other. Joining is the problem type that Greeno (Note 3) identifies as a change situation in which the direction of change is an increase. Figure 2 demonstrates where the joining occurs in the three-dimensional matrix we have described. We present some examples.

Example 6. Wally has 3 pennies. His father gives him 6 more pennies. How many pennies does he have altogether? [Open sentence: $3 + 6 = \square$.]

Example 7. Marcia walked 7 blocks. Then she ran some more blocks. Altogether she walked and ran 12 blocks. How many blocks did she run? [Open sentence: $7 + \square = 12$.]

Example 8. John is 5 years old. How old will he be 8 years from now? [Open sentence: $5 + 8 = \square$.]

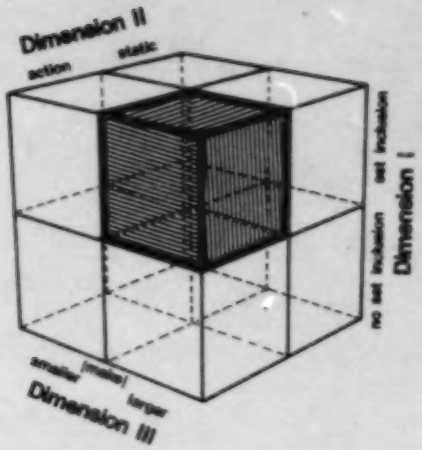

Figure 2. Schematic representation of "joining" problem

Separating

Like joining, separating is a familiar situation to young children and, as such, appears frequently in most elementary mathematics instructional programs. Separating is the process of breaking up a single entity A into two subentities B and C, and then removing one of those two. Prior to separating a larger entity A into its two component subentities B and C, those two subentities are not distinguishable from each other. This is similar to the joining process. Greeno (Note 3) identifies the change direction as being a decrease. Gibb (1956) describes this problem as a "subtractive" situation. Figure 3 locates separating within the matrix. Here are some examples.

Example 9. Fred has some candies. He gives 3 of them to Kathy. Now he has 4 candies left. How many candies did he have to begin with? [Open sentence: ☐ - 3 = 4.]

Example 10. Sue has 14 pennies. She spent 11 pennies on a toy. How many pennies does she have left? [Open sentence: 14 - 11 = ☐ .]

Example 11. Maria has a board that is 17 cm long. She uses a piece 8 cm long from that board to make a support for a race track. How long is the piece of board that is left? [Open sentence: 17 - 8 = ☐ .]

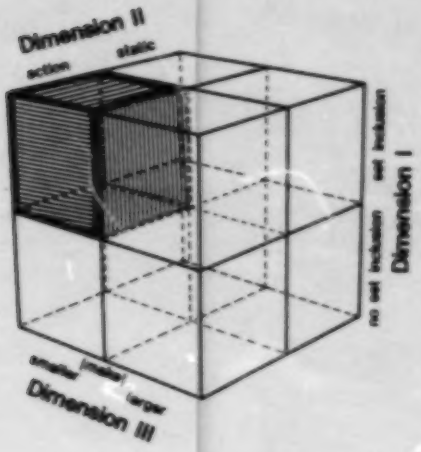

Figure 3. Schematic representation of "separating" problem.

Equalizing

Equalizing is a problem type that is not as well-known as joining and separating. It is used extensively in the earlier sections of the Developing Mathematical Processes (DMP) program that was developed at the Wisconsin R & D Center for Cognitive Learning (Romberg et al., 1974). Equalizing is the process of changing one of two entities, so

that the two are then equal or the same on some particular attribute. Consider two disjoint entities A and C so that $a \overset{<}{-} c$. Then some entity B is adjoined to A so that A $\cup$ B = C'; i.e., the union C' of A and B is such that it is equal in measure to the given entity C.

This situation is known as an equalize-add on problem. A definite or implied action is carried out that is incremental in nature. That is, the action has the effect of making entity or set A larger. The location of the equalize-add on problem within the matrix is shown in Figure 4. Here are some examples.

Example 12. There are 6 boys and 9 girls on the class basketball team. How many more boys have to be put on the team so there will be the same number of boys and girls? [Open sentence: $6 + \square = 9$.]

Example 13. There were 11 cups on the table. Then I put 5 more cups on the table so there would be just as many cups as there were saucers on the table. How many saucers were on the table? [Open sentence: $11 + 5 = \square$.]

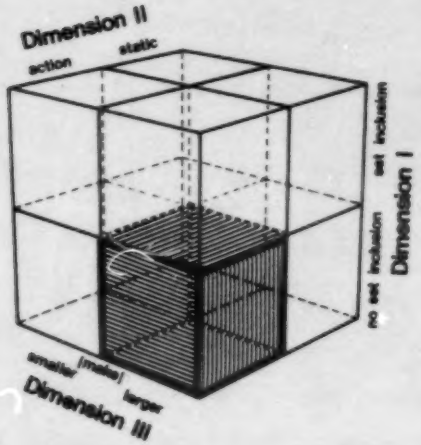

Figure 4. Schematic representation of "equalize-add on" problem.

A different equalizing situation occurs when $a \overset{>}{-} c$, in which case a subset B is separated from A so that the remaining set C' is equal in measure to C. This situation is shown as an equalize-take away problem. Here the definite or implied action is decremental in nature. Entity or set A is made smaller in order to equalize it with C. Figure 5 shows the location of equalize-take away problems in the matrix. Some examples follow.

Example 14. There were 10 chairs at the table. Then 2 chairs were taken away so that there would be the same number of chairs as there were people to sit at the table. How many people were there to sit down? [Open sentence: $10 - 8 = \square$.]

Example 15. I have a piece of yarn that is 13 units long. How much do I have to cut off so that the yarn will be as long as this board which is 6 units long? [Open sentence: $13 - \square = 6$.]

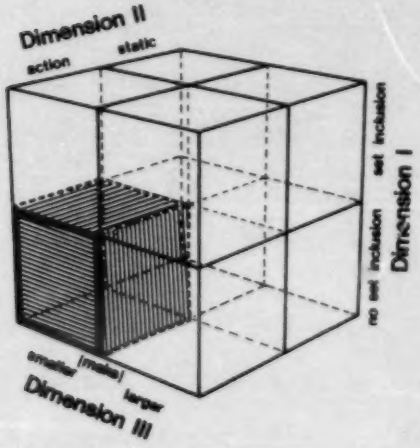

Figure 5. Schematic representation of "equalize-take away" problem.

Part-Part-Whole

This is a static relationship that exists between an entity having a particular attribute and its two disjoint, but complementary, parts. This part-part-whole relationship is isomorphic to the joining action in that both have the same set-notation representation, namely, $A \cup B = C$. Part-part-whole is different from joining in that its two subentities will have some identifying characteristics that distinguish them from each other regardless of their position or any other relationship between them. Steffe (1970) and Le Blanc (1971) say there is no described transformation; Greeno (Note 3) describes this type of problem situation as a Combination schema; Nesher and Katriel (Note 4) use the term Static description.

We make note of the fact that the part-part-whole problem type--a static situation in which the set-inclusion relationship is present--does not appear to divide itself naturally along the third dimension of order. It is tempting to try to dichotomize the problem types on some logical basis. However, such attempts have proven fruitless. Yet, it is interesting to conjecture that at some future time, empirical evidence dealing with the processes children use to solve the various kinds of part-part-whole problems may well indicate a natural partitioning of the problems along the order dimension. For the present, we have decided to have part-part-whole problems occupy two cells in the matrix, as shown in Figure 6. Some examples of part-part-whole problems follow.

Example 16. There are 2 boys and 6 girls in the dancing class. How many children are there altogether? [Open sentence: 2 + 6 = ☐ .]

Example 17. Maria has some toy cars that are either blue or red. She has 4 red cars and 9 cars altogether. How many blue cars does she have? [Open sentence: 4 + ☐ = 9 or ☐ + 4 = 9.]

Example 18. Victor spends 25 minutes doing math and science homework. If he takes 15 minutes to do his math, how much time does he spend doing science? [Open Sentence: 15 + ☐ = 25 or ☐ + 15 = 25.]

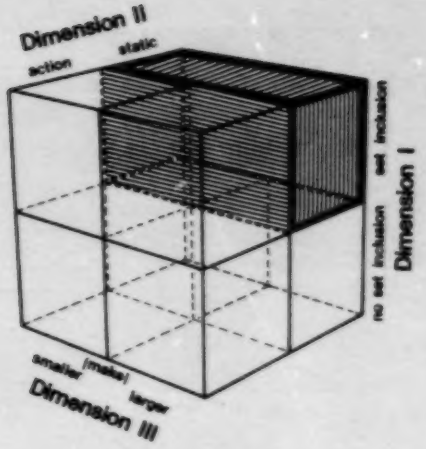

Figure 6. Schematic representation of "part-part-whole" problems.

Comparison

The static relationship of order existing between two disjoint entities is usually called the comparison (Gibb, 1956; Greeno, Note 3; Nesher & Katriel, Note 4) or difference situation. A comparison-larger problem situation involves the amount by which the larger of two compared entities exceeds the smaller on a stipulated attribute. In this situation, the student is directed by the problem language to focus on the fact that some entity A is larger than a second entity B. The two entities being compared are disjoint. The amount of difference or excess of A over B is c. The number c is the measure of an entity C that is a subset of A. The relationship among A, B, and C is $A = C \cup B'$ where B' is an entity equal in measure to the given entity B. Figure 7 indicates the location of comparison-larger problems in the matrix. We present several examples.

Example 19. Mark has 2 balloons. His sister Connie has 7 balloons. How many more balloons does Connie have than Mark? [Open sentence: $2 + \square = 7$ or $7 - 2 = \square$.]

Example 20. Joe has 13 records. Mike has 8 more records than Joe. How many records does Mike have? [Open sentence: $13 + 8 = \square$.]

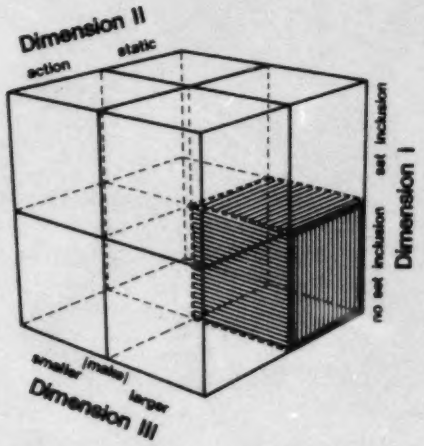

Figure 7. Schematic representation of "comparison-larger" problem.

In example 19, the choice of two possible open sentences was dictated by reason of empirical evidence (Carpenter, Hiebert, & Moser, Note 2). In that study, a considerable number of children used a solution strategy for a comparison-larger problem which is best represented by the open sentence $a - b = \square$.

A second, similar type of problem is called the comparison-smaller problem. As with the comparison-larger problem, it involves a static relationship of order between two disjoint entities. However, the comparison-smaller problem focuses attention on the smaller of the two entities and on the amount by which it is less than the second entity. The location of comparison-smaller problems in the problem matrix is shown in Figure 8. We present some examples.

Example 21. A rock weighs 13 units. This brick weighs 19 units. How much lighter is the rock than the brick? [Open sentence: 19 - 13 = $\square$ or 19 - $\square$ = 13.]

Example 22. Mary has 7 cherries. Louisa has 11 peaches. How many fewer pieces of fruit does Mary have than Louisa? [Open sentence: 11 - 7 = $\square$ or 11 - $\square$ = 7.]

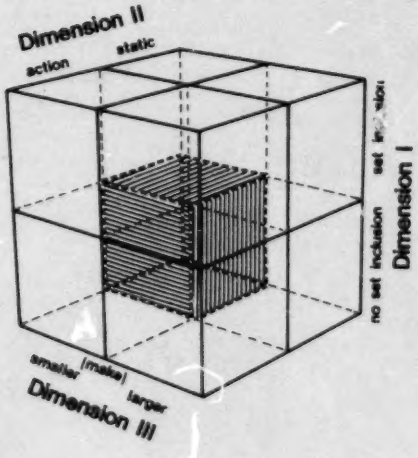

Figure 8. Schematic representation of "comparison-smaller" problem.

In presenting examples of the various types of problem situations, a deliberate attempt was made to include several noncanonical examples. That is, there were instances where the problem was stated in such a way so that in the mathematical model (the sentences $a + b = c$ or $a - b = c$), the missing number would not always be in the "c" position. This is a characterization of the missing third number in a problem, the third problem component to which we now direct our attention.

Third problem component: Characteristic of the missing number. In the preceding section, a deliberate attempt was made to include several examples of noncanonical examples as well as canonical ones within each problem type. This particular distinction between canonical and noncanonical is precisely the way in which we have chosen to characterize the third problem component--the missing number.

A canonical situation is one in which the literal mathematical sentence representation would be a canonical sentence of the form $a + b = \square$ or $a - b = \square$. As an example, consider the joining problem, "Wally has 3 pennies. His father gives him 6 more pennies. How many pennies does he have altogether?" The model for this problem is the canonical addition sentence $3 + 6 = \square$ and the expectation is that the problem solver would impose an additive structure to the problem. On the other hand, if the conditions of the problem were changed slightly to, "Wally has 3 pennies. His father gave him some more pennies. Now he has 9 pennies altogether. How many pennies did his father give him?," the direct literal translation or representation

would be the noncanonical sentence 3 + ☐ = 9. This joining problem has now taken on a totally different structure from the first joining problem even though both involve the action of joining and both involve the number triplet 3, 6, 9. The second problem is an exemplar of the "missing-addend" situation which has been traditionally thought of as a particular type of subtraction problem.

It is important to note that the suggestion of canonical or noncanonical sentence models is one that we have adopted to make clear the distinction between types of problem situations. It should not be inferred that we expect young children to choose this particular symbolic modality to represent problems. In fact, for the missing-addend type, a more mature child may correctly perceive the situation as one that is amenable to symbolic solution by direct subtraction. This would be particularly true if the numbers in the problem were larger than those given in the above example. Representing the total problem by the sentence 26 + ☐ = 53 might not be as efficient as using 53 - 26 = ☐, especially if the latter were written in the alternative vertical form. Note, however, that using the latter sentence would no doubt involve some type of transformation on the student's part.

Problem structure. In the preceding paragraphs, mention was made of problem structure. Since student interpretation of problem structure plays such a major role in determining how that student attempts to solve the problem, i.e., what sequence of representations and

transformations he or she chooses, it is important that we attempt to clarify what is meant by problem structure. We suggest that it is the combination of the second and third problem components discussed earlier which gives a problem its structure.

A strictly mathematical interpretation can be given to a problem structure. Looking momentarily at symbolic problems, component two--operation--indicates there are two possibilities, addition and subtraction.

Addition: $a + b = c$ Subtraction: $a - b = c$

When the third problem component--location of missing number--is considered, then the structure becomes more apparent. For the sake of the example, suppose the missing number is in the "a" position, giving

Addition: $\square + b = c$ Subtraction: $\square - b = c$.

The mathematical structure is simply defined as what binary operation on the two remaining numbers is required to give the correct solution. Or thinking in another fashion, if the ordered pair (c,b) was entered into a Reverse Polish Notation (RPN) calculator, which operation key should be pushed to get the correct answer? Regardless of the question, the answer for the sentence labeled "Addition: $\square + b = c$" is subtraction, which is to say that the problem has a subtractive structure, mathematically speaking. For the second sentence labeled "Subtraction: $\square - b = c$," the answer is addition which means it has an additive structure.

For a pure mathematician the consequence of this analysis for instruction would be abundantly simple. That is, teach children to look at or analyze a problem situation in terms of deciding which binary operation to perform on the two given numbers, and once the correct analysis is made, to carry out that operation. However, anyone who has worked with young children knows that things are not that simple. Children are not, and do not necessarily think like mathematicians or even like mature adults who are not mathematicians. Children interpret problems within the limitations of existing mental capabilities. Thus, it seems that we must also consider a psychological structure of problems and the degree to which it maps directly on the problem's mathematical structure.

It is important to recognize that young children do not have a firm concpetual understanding of what addition and subtraction are. At the stages being considered in this paper, these concepts are emerging and developing in the children. Children impose a psychological structure upon addition/subtraction problems as evidenced by the way they attempt to solve those problems. This may indicate at what stage their conceptual development is.

Consider as examples the following two problems that are both subtractive situations when viewed from the mathematical point of view. The first is a canonical separating situation in which 5 apples are to be given away from a set of 9 apples. The overwhelming tendency of young children is to view this problem as a "take-away" situation in

which, if the apples are represented by physical models, an active decrementing procedure is used to actually take 5 objects away from the set of 11 objects. When physical models are not used, children ordinarily count down from 11. This is a verbal analogue to taking away. The second example is a noncanonical joining situation in which an unknown set of pennies has been added to a set of 3 pennies to give a total set of 9 pennies. Here some children will use a "take-away" strategy as described above, but a sizeable number will employ an "add-on" procedure. In this case, the set of 3 is set out and then added to incrementally one-by-one until the set of 9 is reached, at which time the number of objects or counting words used in the incrementing is determined as the answer. In the sense that these problems produced differing behaviors, we propose that they have different psychological structures.

In effect, we are suggesting that the psychological structure of addition and subtraction problems is a relative matter. For many of the variations--canonical vs. noncanonical--of the different problem types that were represented in the three-dimensional two-by-two-by-two matrix, the psychological structure appears to be unambiguous in that almost all children treat those problems in the same way. For example, the canonical joining problem alluded to earlier about Wally and his pennies maps almost perfectly onto the mathematical canonical addition problem. This is true whether it is Wally or Fred or Mary or whether the objects are pennies, candies, or pet frogs or whether the number triplet is (3,6,9), (5,7,12), or (2,8,10).

For other problem situations, the psychological structure imposed by children is not so obvious. Furthermore, one can reasonably expect the child's imposition of structure to change as he or she changes developmentally and as the result of direct instruction. It is also possible that various semantic differences in the statement of nonsymbolic problems could produce some differing interpretations by children. So, too, could relative number size. For example, in a noncanonical part-part-whole situation involving the number triplet (1,8,9), the use of the given numbers 1 and 9 might initiate a different problem-solving response than use of the numbers 8 and 9. For symbolic problems, researchers such as Groen and Poll (1973) suggest that, in fact, differing strategies are used. In general, we propose that a great deal of further investigation is indicated in the area of children's imposition of psychological structure on addition and subtraction problems.

Conclusion

A major goal of research in mathematics education is a better understanding of how children acquire concepts and skills related to mathematics. An important area of interest is arithmetical learning, and, in particular, the operations of addition and subtraction. One method of gaining information about children's understanding and use of addition and subtraction is to investigate the factors associated with their ability to solve selected addition and subtraction problems.

When this knowledge is combined with other knowledge about children's intellectual and cognitive development, about their understanding of number itself, one may then begin to propose ways of adjusting instruction to accommodate the tremendous variability that exists among children.

In this paper we have set out some definitions dealing with one aspect of solving addition and subtraction problems. Part of the solution process involves representation. Representation has been defined as the change from one problem modality to another. The various types of modalities have been discussed and exemplified. And, finally, the components of a problem have been characterized together with the two main types of problems themselves. In the other documents to be produced by the Mathematics Work Group of the Wisconsin Research and Development Center for Individualized Schooling, the terms introduced and described in this paper will be used consistently.

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